Role of Flexible Functional Form in Estimating Local Government Structure

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Abstract

This paper examines the relationship between population heterogeneity and decentralized local government structure without making any a-priori assumption of the estimating equation and making use of consistent data over time. Past studies assume specific functional forms and fail to control for consistency in the pooled data. A second-order Taylor's series expansion of a transcendental logarithmic function is applied to estimate the nature of the functional form of the local government structure. Pooled data from a consistent sample of 95 major metropolitan areas in the U.S. during 1982-1992 are employed, and several hypotheses are tested. The function is restricted according to the hypotheses test-results and then estimated as the optimal functional form. Unlike past studies, the results show strong linear trends of the population heterogeneity variables for special districts, but strong non-linear trends for other types of local governments.

I. Introduction

In the regional science literature, empirical studies examining the relationship between local government structure and population characteristics tend to find that greater heterogeneity in population characteristics within a region is associated with a more decentralized local government structure. Researchers (Nelson, 1990; Kenny and Schmidt, 1994; Fisher and Wassmer, 1998; and Wassmer and Fisher, 2000) believe that variation in population characteristics such as income and age reflects variation in demands for locally provided government services. In regions where the population is more heterogeneous, mobile residents will sort themselves into larger numbers of relatively smaller jurisdictions so they reside among others with similar public service demands. However, these studies generally make a priori restrictions on the estimating equation and fail to control for consistency in the data over time. The studies that make use of data from more than one time period, the definitions of metropolitan areas do not appear to remain constant over time making it harder to draw conclusions about the relationship between changing population characteristics and government structure.

In this paper, a second-order Taylor's series expansion of a translog function is proposed to estimate the functional form of the local government structure, which is represented by the number of jurisdictions. An advantage of using a second-order Taylor's series expansion of a translog function is that it does not impose any a priori restrictions on the estimating equation (Christensen et al., 1975). We conduct various specification tests of the estimating function to examine the functional relationship between the population heterogeneity variables and the number of jurisdictions for each type of local government structure. Then we examine the relationship between population heterogeneity and decentralized local government structure within metropolitan areas in the U.S. using data from different time periods and employing consistent definitions of regional boundaries over time.

Therefore, the main contribution of this paper is to explore the nature of the functional form of the local government structure. The paper is organized as follows. We briefly review the past literature in the next section. In Section III we develop an empirical model for the local government structure and formulate various hypotheses that will be tested. In Section IV, we present the empirical results and provide conclusion and future research directions in the final section.

II. Literature Review

The first theoretical literature on population heterogeneity in regional science began with the work of Tiebout (1956). The Tiebout model argues that local governments compete for residents by offering bundles of public goods. Residents then sort themselves among jurisdictions according to their income and tastes/preferences. Following Tiebout, Musgrave (1959) argued that welfare gains are realized when local public goods are supplied based on the particular tastes and circumstances of different jurisdictions.

In 1972, Oates extended and generalized Tiebout's model by using the fiscal federalism approach, which is the decentralized provision of public goods to improve efficiency. According to Oates, decentralization can increase efficiency by improving the match between consumers' demands and services provided by the governments in the absence of mobility of consumers, economies of scale, and intergovernmental externalities associated with the provision of any particular provision of government service. Oates supported Tiebout's assumption that all income takes the form of dividend income. But, according to Oates, this assumption breaks any link between the jurisdiction of residence and the jurisdiction in which one works. However, based on Oates, this assumption makes more sense for analysis in a metropolitan area wetting with a multiplicity of small governments where workers can reside in one community and live in another.

Another problem in the Tiebout model is the assumption of identical production functions for local services. This assumption is less compelling in the public sector if different degrees of population heterogeneity exist across metropolitan areas. Schwab and Oates (1991) found that although communities may have similar access to means of production and even produce the same level of "direct output" such as police patrols and school classes, they have less control over the "indirect output" or the level of final services, such as the degree of safety in the community or the quality of the local school system. The level of final services not only depends on the means of production but also on the community composition. This finding shows that the supply side of Tieboout's model is also important and that population heterogeneity enters into the production function as well.

Moreover, both Brueckner & Lee (1989) and Schwab & Oates (1991) found that in the presence of population heterogeneity within a community, there exist certain gains in production in terms of cost savings from the “mixing” of different types of people. According to them, optimal community composition in a community with different types of people can involve a tradeoff between the cost savings from heterogeneity in production and the gains from homogeneity in consumption.

The fiscal federalism approach to the optimal structure of local governments has been criticized. For example, Hochman et al. (1995) found that the ideas of fiscal federalism (where different layers of governments provide different services) cannot be generalized to analyze the optimal structure of all local governments. Hochman et al. (1995) suggested that consolidation is not supported by the local governments whose residents can free-ride on the public services provided by central cities.

The public choice theorists (Brennan & Buchanan, 1980; Zax, 1989) argued quite differently than Oates. They justify decentralization as a solution to government failures arising from imperfect collective choice mechanisms or bureaucratic monopoly. Population heterogeneity may still exist in communities where individual choice is performed under a collective decision-making rule. However, Buchanan and Tullock (1962) argued that heterogeneous communities have relatively higher external costs of collective decision-making, and therefore, imposition of state laws are more likely to reduce the external costs of collective decision-making in communities with a greater degree of population heterogeneity. In addition, Temple (1996) found that voters in communities with more homogeneous preferences would like to eliminate state laws imposing fiscal limitations.

Even though population heterogeneity and demand for local public bundle received a lot of attention in the past empirical literature, there are only a few empirical studies explaining how population heterogeneity may influence the structure and size of local governments, such as Nelson (1990), Kenny & Schmidt (1994), Martinez-Vazquez et al. (1997), Wassmer & Fisher (1997). These studies examined the relationship between population heterogeneity and the number and types of local governments within a metropolitan area. They have found that local government structure is influenced by economic and demographic characteristics of the population.

Nelson (1990) was the first cross-sectional empirical study on this topic. He concluded that greater heterogeneity of preferences led to more general-purpose and special district units of local governments. The second cross-sectional study was done by Fisher & Wassmer (1997). Their results also suggest that greater differences in the variation of citizen preferences will lead to more (smaller) governments. Kenny & Schmidt (1994) attempted to incorporate the tradeoff between benefits of scale economies costs due to population heterogeneity. Martinez-Vazquez et al. (1997) examined how racial heterogeneity influences the number of school districts and special districts for both state and metropolitan areas in the United States. They found no evidence that racial heterogeneity affected the formation of special districts, but income and age variation variables were positively and statistically significant with respect to school districts. In short, both the theoretical and empirical review of the literature suggest that population characteristics and population heterogeneity do have some influence on the local government structure.

However, in the studies that make use of data from more than one time period, the definitions of metropolitan areas do not appear to remain constant over time making it harder to draw conclusions about the relationship between changing population characteristics and government structure. In this research, we examine the relationship between population heterogeneity and decentralized local government structure within metropolitan areas in the U.S. using data from different time periods and employing consistent definitions of regional boundaries over time. Following the earlier studies cited above, we make use of measures of income and age heterogeneity within metropolitan areas as proxies for variations in residents’ demands for locally-provided government services.

In addition to examining the relationship between regional variation in income and age and local government structure, we pay special attention to the role of racial variation. As in the case of income and age, race has long been used in empirical studies (e.g., Bergstrom and Goodman, 1974) as a predictor of demand for government services. While Nelson (1990) notes that the statistical significance of race in empirical public service demand equations has never been adequately explained, Fisher and Wassmer (1998) use racial heterogeneity as well as income and age heterogeneity to reflect the variation within metropolitan areas in demands for local public services.

III. The Empirical Model

In this section, we derive an empirical model for the local government structure, which can be represented by the number of jurisdictions. We let J be the number of jurisdictions in a metropolitan area and assume that the technology of production is identical within a metropolitan area. We assume that the number of jurisdictions is related with variations in tastes and income factors. The taste factors include age, race, and education. Meanwhile, due to difficulties with using a large number of variables in the analysis, we restrict our focus to the population heterogeneity variables only. Therefore, the empirical model can be shown as follows,

$$J = J(H_Y, H_E, H_A, H_R), \qquad (1)$$

where H_Y, H_E, H_A, H_R stand for variations in income, education, age, and race, respectively.

According to Equation (1), the second-order translog regression model can be derived as,

$$\ln J = \beta_0 + \sum_k \beta_k \ln H_k + \frac{1}{2}\sum_k \sum_l \ln H_k \ln H_l + \varepsilon_t \qquad (2)$$

where $k, l = Y, E, A,$ and R, and the point of approximation is taken to be the unit vector (so that at the point of approximation, $\ln H_K = 0, \forall k$); and is stochastic disturbance terms, assuming a mean 0 and a variance σ^2. It is also assumed that the independent variables are uncorrelated with stochastic disturbance terms. For detail, Equation (2) can be rewritten as,

$$+\frac{1}{2}\beta_{EE}(\ln H_E)^2 + \frac{1}{2}\beta_{AA}(\ln H_A)^2 + \frac{1}{2}\beta_{RR}(\ln H_R)^2 + \beta_{YE} \ln H_Y \ln H_E$$

$$+\beta_{YA}\ln H_Y \ln H_A + \beta_{ER}\ln H_E \ln H_R + \beta_{AR}\ln H_A \ln H_R + \varepsilon_t . \quad (3)$$

Moreover, we formulate the following hypotheses (see appendix for detail of our hypotheses) that will be tested at the point of approximation using the flexible functional form described in Equation (3).

H1: The number of jurisdictions, J, is linear with respect to the population heterogeneity variables.

H2: The number of jurisdictions, J, is weakly additive with respect to the population heterogeneity variables.

H3: The number of jurisdictions, J, is linearly homogenous in population heterogeneity variables.

H4: The population heterogeneity variables are weakly separable.

IV. Empirical Results

In this section, we present the results for the tests of the four main hypotheses formulated in the preceding section regarding the functional form of the local government structure. The Wald test statistic for non-linear restrictions is used to test the various hypotheses.

The Wald test statistic can be shown as follows,

$$W = Wald = \chi^2[J] = [R(c_1) - r]' \{Est.Asy.Var[R(c_1) - r]\}^{-1}[R(c_1 - r)].$$

For any hypothesis to be tested, J is the number of non-linear restrictions on K parameters, θ, in the form H_0: $R(\theta) - r = 0$, where q is the parameter vector being estimated. c_1 is the maximum likelihood estimates of θ estimated without the restrictions, and c_0 is the restricted maximum likelihood estimates.

We employ pooled data from a sample of 95 major metropolitan areas in the U.S. during 1982-1992, and the ordinary least squares (OLS) method is used to estimate Equation (3). Based upon the OLS estimation, those various hypotheses are tested by using the Wald test statistic, and the results of the Wald statistics are reported in Table 1.

As Table 1 shows, except for the special districts, we do not reject the hypothesis that the number of local governments is linearly related with the four population heterogeneity variables. It should be noted that Nelson (1990), while estimating the local government structure, assumed a log-linear form of the local government structure, but others (Martinez-Vazquez et al., 1997; Fisher & Wassmer, 1998) used some non-linear forms of estimation.

In addition, for consistency, we test for strong additivity. If the local government structure is linear, then it must be strongly additive as well. As Table 1 shows, we do not reject strong additivity for special districts, but for all other types of local governments strong additivity are rejected. However, the number of local governments is additive for total governments, municipalities, and school districts.

We reject the hypothesis that the number of local governments is linear homogeneous in the population heterogeneity variables except for special districts. This implies that increases in all the population heterogeneity variables by some fixed proportion will not lead to an increase in the number of local governments by the same proportion. Nevertheless, all types of local governments are additive and homothetic except for the school districts.

With the exception of school districts, all other types of local government structure show that the population heterogeneity variables are weakly separable with respect to all partitioning of those variables. Meanwhile, we test Hypotheses 4.8, 4.9, and 4.10 together and do not reject them, which means that we cannot reject pair-wise weak separability of the population heterogeneity variables. For this reason, Hypotheses 4.1 to 4.6 cannot be rejected.

Furthermore, we re-estimate the functional form of each type of local government based on the results from the Wald test. The estimates of each type of local government after imposing the restrictions are presented in Table 2. As Table 2 shows, the findings of the influence of population heterogeneity variables on local government structure reveal that even though the individual population heterogeneity variables do not strongly support the decentralization hypothesis, the significant effects of interactive population heterogeneity variables support the "heterogeneous demand" argument rather than the "taste for association argument." Each of the population heterogeneity variables may not be significantly affecting the number of local governments directly, but through their interaction the local government structures across metropolitan areas are affected.

Additionally, as Table 2 shows, income heterogeneity is positive and significant for municipalities. Income and age interaction variable is positive and significant for the total number of governments. Age heterogeneity directly affects the special districts too. Also, population diversity in both income and age lead to statistically significant increase in the number of municipalities and total number of governments. Income-race interaction variable positively influences the number of municipalities. Education-race and age-race heterogeneity interaction variables are positive and significantly influence the school districts.

Finally, when we re-estimate the special districts by using the linear form of population heterogeneity variables, the constant term is extremely significant along with income and age heterogeneity. This means that other factors captured by the constant term have significant influence on the local government structure. This is consistent with our earlier findings for both cross-sectional and pooled data. The linear form of the special districts indicates that it is of the Cobb-Douglas form, and thus, has all the properties of a Cobb-Douglas function. It is strongly additive and point-wise separable. Therefore, it is also weakly additive and weakly separable. Consequently, our findings confirm what the theory predicts.

V. Conclusion

In this paper, we address an important issue of the choice of the correct functional form of the estimation equation. Instead of simply estimating a specific functional form, a second-order Taylor's series expansion of a translog function is proposed to estimate the functional form of the local government structure, which is represented by the number of jurisdictions. We estimate the number of jurisdictions in a metropolitan statistical area in the USA as a function of variation in income, education, age, and race. We also formulate several hypotheses, expressed as conditions on the general function. The function is restricted according to the hypotheses test-results and then estimated and presented as the optimal functional form for estimating the research question.

As a result, the linearity test shows that there are strong linear trends of the population heterogeneity variables for special districts, but strong non-linear trends for other types of local governments during 1982-92. Therefore, any conclusive evidence of the decentralization hypothesis depends on the correct linearity or non-linearity form of the estimating equation. Also, when the estimating equation is non-linear, some of the population heterogeneity variables are weakly separable with respect to some partitioning of those variables while others are not weakly separable. Thus, when the interactive population heterogeneity variables are included in estimating the local government structure, the interaction terms are not just a combination of only two but three population heterogeneity variables.

Finally, results indicate that a proper specification of the local government structure not only is important but also complex. This study suggests that a more general form of the estimating equation is appropriate for all types of local governments, except for the special districts.

Table 1: The Wald Test Statistic for Various Hypotheses for the Number of Local Governments in 95 Major U.S. Metropolitan Areas during 1982-1992

Hypothesis	Total Governments	Municipalities	School Districts	Special Districts
Linearity	22.46**	36.12***	28.79***	11.81
Strong Additivity	13.70**	65.92***	17.90***	4.81
Additivity	1.14	2.11	1.65	0.32
Linear Homogeneity	10.05*	13.64**	19.07***	2.82
Additivity & Homotheticity	10.78	8.76	15.85**	NA
Weak Separability of population heterogeneity variables Hypothesis 4.8, 4.9, and 4.10 tested together	7.31	5.34	15.95***	NA

N=190. Wald statistic (Chi-squared values) with *** (99%), ** (95%), and * (90%) confidence, respectively. NA denotes not applicable.

Table 2: Pooled Cross-sectional Ordinary Least Squares Regression Estimates for the Number of Local Governments in 95 Major U.S. Metropolitan Areas during 1982-1992

Explanatory variable	Total Governments	Municipalities	School Districts	Special Districts
Heterogeneity measures				
Income	85.75 (1.61)	197.91 (3.39)***	82.83 (1.07)	-2.75 (-3.51)***
Income-squared	-16.59 (-1.64)	-36.28 (3.29)***	-15.03 (-1.03)	NA
Education	3.49 (0.08)	-32.48 (-0.67)	8.64 (0.14)	0.66 (0.94)
Education-squared	-8.48 (-0.97)	-9.08 (-0.95)	-15.17 (-1.20)	NA
Age	-125.93 (-1.95)*	-244.08 (-3.45)***	-164.93 (-1.76)*	3.19 (3.12)***
Age-squared	-41.07 (-2.65)***	-63.96 (-3.77)***	-59.58 (-2.66)***	NA
Race	-0.91 (-0.17)	-12.13 (-2.10)**	1.52 (0.20)	0.04 (0.45)
Race-squared	0.13 (0.71)	-0.20 (-0.99)	0.19 (0.73)	NA
Income* Education	-3.55 (-0.42)	4.31 (0.47)	-5.51 (-0.45)	NA
Income * Age	20.43 (1.79)*	40.94 (3.27)***	25.51 (1.54)	NA
Income * Race	0.84 (0.77)	2.68 (2.25)**	1.11 (0.70)	NA
Education* Age	-6.80 (-0.71)	-10.11 (-0.97)	-7.59 (-0.55)	NA
Education* Race	0.22 (0.27)	0.24 (0.26)	2.21 (1.87)*	NA
Age * Race	2.61 (2.08)**	0.14 (0.10)	4.39 (2.42)**	NA
Constant	-230.06 (-1.57)	-549.78 (-3.42)***	-241.18 (-1.13)	18.82 (4.54)***
R-squared	0.21	0.36	0.18	0.08

N=190. t-statistics in parentheses with ***(99%), **(95%), and *(90%) confidence, respectively. NA denotes not applicable.

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Appendix

(Formulation of Hypotheses and Parameter Restrictions)

Hypothesis 1: The number of jurisdictions, J, is linear with respect to the population heterogeneity variables.

If J is linear, the second-order terms in Equation (3) are zero. *i.e.* $\beta_{kl} = 0 \ \forall \ k,l \in Y, E, A, R$.. This will yield ten independent restrictions.

Hypothesis 2: The number of jurisdictions, J, is weakly additive with respect to the population heterogeneity variables.

If J is weakly additive then, $\ln J = F\,(\Sigma f_k\,(H_k))$, (4)

where $k \in Y, E, A$, and R \ln $J = F\,[f_Y\,(H_Y) + f_E\,(H_E) + f_A\,(H_A) + f_R\,(H_R)]$.

At the point of approximation,

(a)

$k = Y, E, A, R$

(b) $\frac{\partial lnJ}{\partial lnH_k} = F' \frac{\partial f_k}{\partial lnH_k} = \ \beta_k$

$k \neq l$ and $k, l = Y, E, A, R$

From (a) and (b)

$k \neq l$

or $\beta_{kl} = q\ b_k\ b_l$ $\qquad k \neq l$

where $\theta = \frac{F''}{F'^2}$

Therefore, the parameter restrictions are

(i) $\beta_{YE} = \theta\,\beta_Y\beta_E$

(ii) $\beta_{YA} = \theta\,\beta_Y\beta_A$

(iii) $\beta_{YR} = \theta\,\beta_Y\beta_R$

(iv) $\beta_{EA} = \theta\,\beta_E\beta_A$

(v) $\beta_{ER} = \theta\,\beta_E\beta_R$

(vi) $\beta_{AR} = \theta\,\beta_A\beta_R$

Note that this yields five independent restrictions.

Strong additivity holds if $ln\ J = F\,(\sum f_k\,(H_k)) = \sum f_k\,(H_k) \therefore F' = 1, F'' = 0 \Rightarrow \theta = 0$

Therefore, to test for strong additivty six independent restrictions are to be imposed.

Hypothesis 3: The number of jurisdictions, J is linearly homogenous in population heterogeneity variables.

If J is homothetic, then

$lnJ = f\,[ln\,F\,(H_Y, H_E, H_A, H_R)]$ (5)

where F is homogeneous of degree 1.

Differentiating (5) yields

$$\frac{\partial lnJ}{\partial lnH_k} = f'\,\frac{\partial lnF}{\partial lnH_k} \qquad (6)$$

(7)

From (7)

$$\sum_k \frac{\partial^2 lnJ}{\partial lnH_k \partial lnH_l} = f' \sum_k \frac{\partial^2 lnF}{\partial lnH_k \partial lnH_l} + f''\,\frac{\partial lnF}{\partial lnH_l} \sum_k \frac{\partial lnF}{\partial lnH_k}$$

$$= \frac{f''}{f'} \frac{\partial lnJ}{\partial lnH_l}$$

$$= f''\,\frac{\partial lnF}{\partial lnH_l}$$

Evaluating at the point of approximation

$$\sum_k \beta_{kl} = \beta_l \gamma$$

$$(\text{where } \gamma = \frac{f''}{f'}, \text{ a constant}) \qquad (8)$$

Therefore,

$$(9)$$

$$\begin{aligned} &\beta_{EY} + \beta_{EE} + \beta_{EA} + \beta_{ER} = \gamma\beta_E \\ &\Rightarrow \beta_{EE} = \gamma\beta_E - \beta_{YE} - \beta_{EA} - \beta_{ER} \end{aligned} \qquad (10)$$

$$\begin{aligned} &\beta_{AY} + \beta_{AE} + \beta_{AA} + \beta_{AR} = \gamma\beta_A \\ &\Rightarrow \beta_{AA} = \gamma\beta_A - \beta_{YA} - \beta_{EA} - \beta_{AR} \end{aligned} \qquad (11)$$

$$\begin{aligned} &\beta_{RY} + \beta_{RE} + \beta_{RA} + \beta_{RR} = \gamma\beta_R \\ &\Rightarrow \beta_{RR} = \gamma\beta_R - \beta_{YR} - \beta_{ER} - \beta_{AR} \end{aligned} \qquad (12)$$

(9) to (12) yields three independent restrictions.

If J is linear homogeneous in the population heterogeneity variables, then

$$\sum_k \frac{\partial J}{\partial H_k}\frac{H_k}{J} = 1$$

$$\Rightarrow \sum_k \frac{\partial^2 \ln J}{\partial \ln H_k \partial \ln H_l} = 0$$

$$\Rightarrow \sum_k \beta_{kl} = 0 \quad \forall\ k, l \qquad (13)$$

Substituting $\gamma = 0$ into (9) through (12) yields the four independent restrictions, implied by linear homogeneity of J.

$$\beta_{YY} + \beta_{YE} + \beta_{YA} + \beta_{YR} = \gamma\beta_Y$$

$$\Rightarrow \sum_k \frac{\partial \ln J}{\partial \ln H_k} = 1$$

$$\Rightarrow \beta_{YY} = \gamma\beta_Y - \beta_{YE} - \beta_{YA} - \beta_{YR}$$

Hypothesis 4: The population heterogeneity variables are weakly separable.

Two types of weak separability can be hypothesized.

(a) pair-wise separability, i.e. *(i - j) - k , l* separability

(b) *(i - j - k) - l* separability, where $i, j, k, l = H_Y, H_E, H_A, H_R$

Weak separability of H_k and H_l from H_m holds if and only if

$$\frac{\partial\left(\dfrac{\partial J / \partial H_k}{\partial J / \partial H_l}\right)}{\partial H_m} = 0$$

This requirement may be manipulated to yield

$$\frac{\partial\left(\dfrac{\partial \ln J / \partial \ln H_k}{\partial \ln J / \partial \ln H_l}\right)}{\partial \ln H_m} = 0$$

This implies

$$\frac{1}{\left(\partial \ln J / \partial \ln H_l\right)^2}\left[\frac{\partial \ln J}{\partial \ln H_l}\frac{\partial^2 \ln J}{\partial \ln H_k \partial \ln H_m} - \frac{\partial \ln J}{\partial \ln H_k}\frac{\partial^2 \ln J}{\partial \ln H_l \partial \ln H_m}\right] = 0$$

At the point of expansion, this implies

$$\beta_l \beta_{km} - \beta_k \beta_{lm} = 0 \qquad (14)$$

To test for pair-wise separability the following hypothesis are formulated.

Hypothesis 4.1: H_Y and H_E are weakly separable from H_A and H_R.

$\therefore\ \beta_Y\ \beta_{EA} = \beta_E\ \beta_{YA}$

and $\beta_Y\ \beta_{ER} = \beta_E\ \beta_{YR}$

or $\beta_{EA} = \quad \beta_{YA}$ (15)

and $\beta_{ER} = \quad \beta_{YR}$ (16)

(15) and (16) are two independent restrictions.

Hypothesis 4.2: H_Y and H_A are weakly separable from H_E and H_R.

$\therefore \quad \beta_Y \ \beta_{AE} = \beta_A \beta_{YE}$

and $\beta_Y \beta_{AR} = \beta_A \beta_{YR}$

or $\beta_{AE} = \quad \beta_{YE}$ (17)

and $\beta_{AR} = \quad \beta_{YR}$ (18)

(17) and (18) are two independent restrictions.

Hypothesis 4.3: H_Y and H_R are weakly separable from H_E and H_A.

$\therefore \quad \beta_Y \ \beta_{RE} = \beta_R \beta_{YE}$

and $\beta_Y \beta_{RA} = \beta_R \beta_{YA}$

or $\beta_{RE} = \quad \beta_{YE}$ (19)

and $\beta_{RA} = \quad \beta_{YA}$ (20)

(19) and (20) are two independent restrictions.

Hypothesis 4.4: H_E and H_A are weakly separable from H_Y and H_R.

$\therefore \quad \beta_E \ \beta_{AY} = \beta_A \beta_{EY}$

and $\beta_E \beta_{AR} = \beta_A \beta_{ER}$

or $\beta_{AY} = \quad \beta_{EY}$ (21)

and $\beta_{AR} = \quad \beta_{ER}$ (22)

(21) and (22) are two independent restrictions.

Hypothesis 4.5: H_E and H_R are weakly separable from H_Y and H_A.

$\therefore \quad \beta_E \ \beta_{RY} = \beta_R \beta_{EY}$

and $\beta_E \beta_{RA} = \beta_R \beta_{EA}$

or$\beta_{RY} = \quad \beta_{EY}$ (23)

and $\beta_{RA} = \quad \beta_{EA}$ (24)

(23) and (24) are two independent restrictions.

Hypothesis 4.6: H_A and H_R are weakly separable from H_Y and H_E.

$\therefore \quad \beta_A \ \beta_{RY} = \beta_R \beta_{AY}$

and $\beta_A \beta_{RE} = \beta_R \beta_{AE}$

or $\beta_{RY} = \quad \beta_{AY}$ (25)

and $\beta_{RE} = \quad \beta_{AE}$ (26)

(25) and (26) are two independent restrictions.

Using (14), *(i-j-k) - l* separability can be tested.

Hypothesis 4.7: H_Y, H_E, H_A are weakly separable from H_R.

$\therefore \quad \beta_Y \ \beta_{ER} = \beta_E \ \beta_{YR}$

$\beta_A \ \beta_{ER} = \beta_E \ \beta_{AR}$

$\beta_Y \ \beta_{AR} = \beta_A \ \beta_{YR}$

or $\beta_{ER} = \quad \beta_{YR}$ *(27)*

and $\beta_{AR} = \quad \beta_{YR}$ (28)

(27) and (28) imply $\beta_{ER} = \quad \beta_{AR}$.

Therefore, (27) and (28) are the two independent restrictions.

Hypothesis 4.8: H_Y, H_E, H_R are weakly separable from H_A.

$\therefore \quad \beta_Y \; \beta_{EA} = \beta_E \, \beta_{YA}$

$\beta_R \; \beta_{EA} = \beta_E \, \beta_{RA}$

$\beta_Y \; \beta_{RA} = \beta_R \, \beta_{YA}$

or $\beta_{EA} = \quad \beta_{YA}$ (29)

and $\beta_{RA} = \quad \beta_{YA}$ (30)

Similarly, (29) and (30) are the two independent restrictions.

Hypothesis 4.9: H_Y, H_A, H_R are weakly separable from H_E.

$\therefore \quad \beta_Y \; \beta_{AE} = \beta_A \, \beta_{YE}$

$\beta_R \; \beta_{AE} = \beta_A \, \beta_{RE}$

$\beta_Y \; \beta_{RE} = \beta_R \, \beta_{YE}$

or $\beta_{AE} = \quad \beta_{YE}$ (31)

and $\beta_{RE} = \quad \beta_{YE}$ (32)

Similarly, (31) and (32) are the two independent restrictions.

Hypothesis 4.10: H_E, H_A, H_R are weakly separable from H_Y.

$\therefore \quad \beta_E \; \beta_{AY} = \beta_A \, \beta_{EY}$

$\beta_R \; \beta_{AY} = \beta_A \, \beta_{RY}$

$\beta_E \; \beta_{RY} = \beta_R \, \beta_{EY}$

$\frac{\beta_R}{\beta_Y}$

or $\beta_{AY} = \quad \beta_{EY}$ (33)

and $\beta_{RY} = \quad \beta_{EY}$ (34)

Similarly, (33) and (34) are the two independent restrictions